Inflation, Interest Rate and Derivatives

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Very preliminary version, please do not quote

Abstract

The purpose of this article is to show how inflation-indexed derivatives could be a useful tool to determine inflation forecasts and how it can be used to improve monetary policy decisions. By studying pricing formulae of various derivatives, we will extract a Forward Inflation Rate that we will compare with Anticipated Inflation Rate to see the efficiency of our model.

1 Introduction

In only a decade inflation-indexed derivatives have known a huge growth in broker markets. The modern inflation indexed market did not start until the late 1980s. In the USA, inflation-linked bonds represents over $1.5 trillion of the international debt market in 2008. In 1997, the United States began to issue Treasury Inflation Protected Securities bonds (TIPS).

Various kind of derivatives are indexed on inflation such as Inflation-Indexed Swaps (IIS). On each payment date, agent A pays the agent B the inflation rate over a defined period, while agent B pays to agent A a fixed rate. There are Zero-Coupon Inflation-Indexed Swaps (ZCISS), Year-on-Year Inflation-Indexed Swaps (YYIIS) and inflation-indexed options like Caplets or Foorlets which are similar respectively to a Call and a Put. But we also use simple options like Put, Call, futures and bonds like TIPS[**?**].

Hughston (1998) is one of the first who made research on Inflation-Indexed derivatives. He has begun with the general theory and the foreign-currency analogy[**?**] called later in our paper FCA. This method compares real rates as the rates in a foreign economy and CPI as an exchange rate between the domestic (nominal) and foreign (real) economies. It has been a standard for modeling inflation-linked derivatives until Jarrow and Yildirim (JY, 2003). They have

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shown a 3-factor Health-Jarrow-Morton (HJM) model for pricing TIPS and Vanilla Inflation Option. A HJM type model is known to be inconvenient for pricing derivatives and it is hard to calibrate this model. Besides, the HJM type model is known to have an issue of (generating) negative interest rate, which certainly has caused concerns in inflation-rate modeling.

The HJM framework refers to a class of models that are derived by directly modelling the dynamics of instantaneous forward-rates. The central insight of this framework is to recognize that there is an explicit relationship between the drift and volatility parameters of the forward-rate dynamics in a no-arbitrage world. The familiar short-rate models can be derived in the HJM framework but in general, however, HJM models are non-Markovian.

These models typically adopt log-normal dynamics for certain observable inflation related variables, for examples, forward price of real zero-coupon bonds[**?**] : JY model with the Hull-White parametrization. Mercurio (2005) has also priced YYIIS, caplets and floorlets with 3 different approachs. He has modeled a JY version of HJM and he has developed 2 market models.

Leung and Wu (2008) has proposed an extended market model consistent to the HJM framework and thus is arbitrage free. Under such a model[**?**], nominal interest rates are guaranteed to be positive, while the inflation rates can be either positive or negative. Furthermore, they show that the dynamics of inflation swap rates can be approximated by displaced diffusion as well and thus they can price inflation swaptions in closed form. Owing to the closed-form formulae for caplets and swaptions, the extended model can be separately calibrated to benchmark derivatives (caps, floors and swaptions[**?**]) on nominal interest rates and inflation rates, using an existing technology developed by Wu (2003) for calibrating the LIBOR market model.

In this article we try to know if we could determine inflation rate using inflation-indexed derivative products. We want to find the forward Inflation which is a forecast of the market inflation rate. We need to compare the results that we will obtain of the forward inflation rate with the anticipated rate which is defined by economists using mood survey and public opinion. We will use these methods to price each derivative, then we will try to determine inflation-index. We will also use the Moreni and Mercurio SABR model. We will then compare our experimental results with datas of inflation rate from the FRED[**?**] and from banks.

We also want to determine if a new Keynesian Phillips curve (NKPC) can be duplicated to compare the Inflation Rate we have calculated and the one estimated by economics' methods. We will calcul the estimated inflation rate with the NKPC standard model and the hybrid one. We will also need to use a calibration for the parameters, we will take the one made by Mercurio and Moreni (2009).

This article is organized as follows: in the next Section we will see a recap about Consumer Price Index, Inflation, Fisher equation and the Taylor rule. Moreover, in the third section we will work on derivative equations of a Zero-Coupon, Year-on-Year, Swaps, Caplet and European call of a TIPS and try to define the Forward Inflation Rate using pricing formulae of derivatives. In a fourth section we will present our results. Then, in a fitfh part we will interpret our results and explain why their are relevant or not. Last Section concludes the research.

2 The Model

2.1 Index and Inflation

Consumer Price Index (CPI) is a monthly-measure that estimates the average price of services and goods bought by a theoretical household. In theory, but also in practice, inflation can become negative. As shown by Kerkhof (2005), market baskets depend of the country. Kazziha (1999) defines the forward CPI at time t as the fixed amount X that is to be exchanged at time T for CPI $I(T)$,

There are 2 approaches for the pricing and the understanding of Inflation indexed Derivatives. The first is the foreign-currency analogy (called FCA as we have said before) from Jarrow and Yildirim (2003) where they assume that nominal and real rates evolve according to correlated one-factor Hull-White models. The CPI can be viewed as the exchange rate between them.

The second approach are market models. One models the evolution of suitable market quantities under some reference measure. Using no-arbitrage pricing theory, we have: $\Gamma_i(t) = E^{T_i}\left[I(T_i) \mid \mathcal{F}_t\right].$

Where E^{T_i} denotes expectation under the Ti-forward risk-adjusted measure Q_{Ti}, with the zero-coupon bond $P(t, T_i)$, and $\mathcal{F}_t$ is the sigma-algebra generated by the relevant market factors up to time t and $\Gamma_i(t)$ the Forward CPI.

The value at time zero of a Ti-forward CPI can be immediately obtained from the market quote $K(T_i)$. In fact $\Gamma_i(0) = I(0)(1 + K(T_i))^i$ where $I(0)$ is the CPI at time 0 and $K(Ti)$ the contract fixed rate.

The previous definition of forward CPI is consistent with the FCA which postulate that the CPI is equivalent to an exchange rate, so it makes sense to define the T_i-forward CPI as

$$\Gamma_i^{FCA}(t) = I(t)\frac{P_r(t, T_i)}{P_n(t, Ti)} \tag{1}$$

where $P_r(t, T_i)$ and $P_n(t, T_i)$ are respectively the real and the nominal discount factor for maturity T_i. Since $I(t)P_r(t, T_i)$ is an asset you can trade in the nominal economy, when we divide it by $P_n(t, T_i)$ we get a martingale under the nominal T_i-forward measure.

Therefore, since $P_r(t, T_i) = P_n(t, T_i) = 1$

$$\Gamma_i^{FCA}(t) = E^{T_i}\left[\Gamma_i^{FCA}(t) \mid \mathcal{F}_t\right] = E^{T_i}\left[I(T_i) \mid \mathcal{F}_t\right] = \Gamma_i(t) \tag{2}$$

The value at time zero of a T_i-forward CPI can be obtained from the market quote $K(T_i)$ of the ZC swap. In fact, see Brigo and Mercurio (2006),

$$\Gamma_i(0) = I(0)(1 + K(T_i))^i \tag{3}$$

The inflation rate of a country is defined in terms of its CPI. Denote by $I(t)$ the CPI at time t,

$$\hat{\imath}(t,T) = \frac{I(T)}{I(t)} - 1 \tag{4}$$

In our study, we will work within the American data. A big part of the index is driven by housing (42%[?]). US CPI is the main inflation index[1]. Inflation rate is the percentage rate of change of the price index over the time period $[t,T]$. For comparison purpose, we will use annualized inflation rate :

$$i(t,T) = \frac{1}{T-t}\left(\frac{I(T)}{I(t)} - 1\right) \tag{5}$$

2.2 Fisher

The Fisher Equation :

$$r_{N(t,T)} = r_R(t,T) + E\left[i(t,T)\right] \tag{6}$$

explains the relation between the nominal interest rates, the real interest rates and the expected inflation rate. Here $r_{N(t,T)}$ and $r_R(t,T)$ denote the cumulative interest rates over the time interval $[t,T]$.

The Fisher equation can be used in either ex-ante (before) or ex-post (after) analysis. Ex-post, it can be used to describe the real purchasing power of a loan. Unlike in ex-ante π is the forecast inflation rate (calculated by econometric models). In Real interest rate from inflation-linked bonds π is the forecast inflation rate calculated by the market using breakeven inflation.

Hence for annual continuous interest and inflation rates, the Fisher equation can be rewritten as follows

$$\exp(r_N - r_R)(T-t) = E\left[\frac{I(T)}{I(t)}\right] \tag{7}$$

2.3 Taylor

The Taylor rule is a monetary-policy rule that stipulates how much the central bank should change its nominal interest rate in response to divergences of actual inflation rate to its target and of actual Gross Domestic Product (GDP) to its potential value[2]. We will choose the following formula which is more general :

$$i_t = \pi_t + r_t^* + a_\pi(\pi_t - \pi_t^*) + a_y(y_t - \bar{y}_t) \tag{8}$$

[1] This index can be found on Bloomberg as CPURNSA.

[2] Often represented by the output with flexible prices. The difference between output and its value with flexible prices is the output gap.

where i_t is the target short-term nominal interest rate (e.g. the federal funds rate in the US), π_t is the inflation rate as measured by the GDP deflator, π_t^* is the desired rate of inflation, r_t^* is the assumed equilibrium real interest rate, y_t is the logarithm of real GDP and $\bar{y}_t$ is the logarithm of potential output, as determined by a linear trend.

3 Options Derivatives

3.1 Zero Coupon

3.1.1 Swaps

A fixed zero-coupon inflation swap (ZCISS) is a bilateral contract that lets a part to secure an inflation-protected return with respect to an inflation index. The inflation buyer pays a predetermined fixed rate $N\left[(1+K)^M - 1\right]$, and receives from the inflation seller inflation-linked a payment $N\left[\dfrac{I(T_M)}{I_0} - 1\right]$.

For the model we will use various methods for each derivatives. Mercurio (2005) prices numerous derivatives using stochastic and non stocastic methods. Using his formula of the price of Zero-Coupon inflation indexed swap (ZCISS)[**?**], we have:

$$ZCISS(t, T_M, I_0, N) = N\left[\frac{I(t)}{I(T_0)}P_r(t, T_M) - P_n(t, T_M)(1+K)^M\right] \tag{9}$$

where $IIC(t, T_M, I, N)$ is the price of the derivative, $I(T)$ the value of CPI at time T, N the nominal value of the contract, K is the contrat fixed rate, the nominal bond $P_n(t, T_M)$ time t price (in dollar) of one dollar in T_M, $P_r(t, T_M)$ is defined as the price at time t and in CPI units of a contract paying one unit of goods and services at T_M, Γ is the forward CPI and T_M is the final time in years.

Knowing that $\Gamma_i(t) = I(t)\dfrac{P_r(t, T_i)}{P_n(t, Ti)}$, we replace in (9) and we obtain:

$$ZCISS(t, T_M, I(T_0), N) = N * P_n(t, T_M)\left[\frac{\Gamma_M(t)}{I(T_0)} - (1+K)^M\right] \tag{10}$$

Dividing (10) by $N * P_n(t, T_M)$ we get:

$$\frac{ZCISS(t, T_M, I(T_0), N)}{N * P_n(t, T_M)} = \frac{\Gamma_M(t)}{I(T_0)} - (1+K)^M \tag{11}$$

Adding $(1+K)^M$ and multiplying by $I(T_0)$ we have:

$$\Gamma_M(t) = I(T_0)\left[\frac{ZCISS(t, T_M, I(T_0), N)}{N * P_n(t, T_M)} + (1+K)^M\right] \tag{12}$$

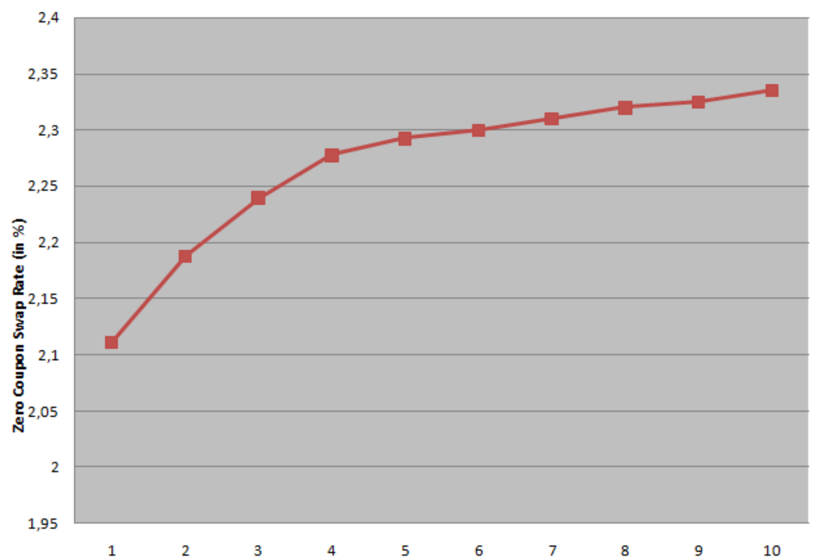

Figure 1: Inflation Swap Rate based on US CPI of date November, 3rd 2004

With the forward CPI, we should find the forward inflation rate. Indeed, calling Forward Inflation Rate

$$\pi(t) = \frac{(\Gamma_{M-1}(t) - \Gamma_M(t)) * 100}{\Gamma_{M-1}(t)} \tag{13}$$

Here we have use the Foreign-Currency Analogy for the forward CPI.

3.2 Year-on-Year

3.2.1 Caplets/Floorlets

An Interest rate cap is a series of european call options or caplets on a specified interest rate. The underlying rate is known as the reference rate. The buyer of the cap receives money if on the maturity of any of the caplets, the reference rate exceeds the agreed strike price of the cap.

We determine the forward CPI from the pricing formula of a caplet. An Inflation-Indexed Caplet (IICplt) is a call option on the inflation rate based on the CPI index. A T_i-maturity caplet (floorlet) is an option on the inflation rate, paying at time T_i

$$\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+ \tag{14}$$

where κ is the strike and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet. In a recent study by Mercurio and Moreni (2009) the pricing of a YY caplet/floorlet[**?**] is noted :

$$IICplt(t, T_{i-1}, T_i, K, \omega) = \omega.P(t, T_i) \left[\begin{array}{c} (1 + \Gamma_i(t)).\phi(\omega d_+) \\ -(1 + K).K.\phi(\omega d_-) \end{array} \right] \tag{15}$$

where

$$\begin{aligned}
d_+ &= \frac{\ln\left(\frac{\widetilde{Y}_i(t)}{K}\right) + \frac{1}{2}\sigma^2(K)(T_i - t)}{\sigma(K)\sqrt{T_i - t}} \\
d_- &= \frac{\ln\left(\frac{\widetilde{Y}_i(t)}{K}\right) - \frac{1}{2}\sigma^2(K)(T_i - t)}{\sigma(K)\sqrt{T_i - t}} \\
\sigma(K) &= \alpha_i \frac{z}{x(z)} \left\{ 1 + \left[\frac{\rho_i \nu_i \alpha_i}{4} + \nu_i^2 \frac{2 - 3\rho_i^2}{24} \right] (T_i - t) \right\} \\
z &= \frac{\nu_i}{\alpha_i} \ln\left(\frac{\widetilde{Y}_i(t)}{K}\right) \\
x(z) &= \ln\left\{ \frac{\sqrt{1 - 2\rho_i z + z^2} + z - \rho_i}{1 - \rho_i} \right\}
\end{aligned}$$

Then to find the Forward Inflation Rate we swap the terms on the right part of the equation and we get :

$$\Gamma_i(t) = \frac{IICplt(t, T_{i-1}, T_i, K, \omega)}{\omega.P(t, T_i).\phi(\omega d_+)} + \frac{(1 + K).K.\phi(\omega d_-)}{\phi(\omega d_+)} - 1 \tag{16}$$

3.2.2 Rate

YY rate can be modeled (analogy of forward LIBOR rates) as on (15) but the problem with this formulation is that YY swap market is not as liquid as the ZC one, explain Mercurio and Moreni (2009). If only ZC swap rates are quoted, it is possible to calculate the initial values $Y_i i(0)$ (by stripping the forward CPI for the quoted maturities), but not the $\Upsilon_i(0)$, which requires the knowledge of YY swap rates.

Having an exact formula for caplet prices does not sufficiently compensate for the loss of an automatic calibration to the inflation linear instruments. We find under the Ti-forward measure: with model parameters $\Upsilon_1(0), \alpha_j, \nu_j, \rho_j$ and $j = 1..M$:

$$\begin{aligned}
\Gamma_i(0) = E^{T_i}\{\Gamma_i(t)\} &= I(0)E^{T_i}\left[\prod_{j=1}^{i}[1 + \Upsilon_i(T_j)]\right] \\
&= I(0)f(\Upsilon_1(0), ..., \Upsilon_i(0), \alpha_1, ..., \alpha_i, \nu_1, ...\nu_i, \rho_1, ..., \rho_i)
\end{aligned} \tag{17}$$

Where f can be calculated by non-standard approximation or Monte Carlo simulation.

3.3 European call on a TIPS

Treasury Inflation-Protected Securities (or TIPS) are the inflation-indexed bonds issued by the U.S. Treasury. The principal is adjusted to the Consumer Price Index, the commonly used measure of inflation. The coupon rate is constant, but generates a different amount of interest when multiplied by the inflation-adjusted principal, thus protecting the holder against inflation. TIPS are currently offered in 5-year, 10-year and 20-year maturities. Thirty-year TIPS did no longer exist.

Jarrow and Yildirim (2000) use a foreign currency analogy, real prices correspond to foreign prices, nominal prices correspond to the domestic prices, and the CPI-U index corresponds to the spot exchange rate. In their research on TIPS, they show how to price an European call[**?**].

The formula of an European Call is

$$
\begin{aligned}
IIC(t,T,I,N) = N.I(t).P_r(t,T_i).&\left(\frac{\log\left(\frac{I(t).P_r(t,T_i)}{K.P_n(t,T_i)}\right)+\frac{\eta^2}{2}}{\eta}\right)\\
&- N.K.P_n(t,T)\left(\frac{\log\left(\frac{I(t).P_r(t,T_i)}{K.P_n(t,T_i)}\right)-\frac{\eta^2}{2}}{\eta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\eta^2 \;=\;& \int_t^T \sigma_n^P(u,T)^2du + 2\int_t^T \rho_{nr}\sigma_n^P(u,T)\sigma_r^P(u,T)du \\
&+\int_t^T \sigma_r^P(u,T)^2du + 2\rho_{nI}\sigma_I\int_t^T \sigma_n^P(u,T)du \\
&-2\rho_{rI}\sigma_I\int_t^T \sigma_r^P(u,T)du + \sigma_I^2(T-t)
\end{aligned}
\tag{18}
$$

where $\sigma_k^P(t,T) = \int_t^T \sigma_k(t,u)du$ for $k = n, r$, $IIC(t,T,I,N)$ is the price of the derivative, $I(T)$ the value of CPI at time T, N the nominal value of the contract, K is the contrat fixed rate, $P_n(t,T_M)$ is the price of a nominal zero-coupon bond at time t maturing at time T_M in dollars, $P_r(t, T_M)$ is the price of a real zero-coupon bond at time t maturing at time T_M in CPI-U units, Γ is the forward CPI and T_M is the final time in years.

Dividing by N and multiplying by η :

$$\frac{IIC(t,T,I,N)*\eta}{N} = I(t)P_r(t,T)\left(\log\left(\frac{I(t)P_r(t,T_i)}{K*P_n(t,T_i)}\right)+\frac{1}{2}\eta^2\right) - K*P_n(t,T)\left(\log\left(\frac{I(t)P_r(t,T_i)}{K*P_n(t,T_i)}\right)-\frac{1}{2}\eta^2\right) \tag{19}$$

Knowing that $\Gamma_i(t) = I(t)\dfrac{P_r(t,T_i)}{P_n(t,Ti)}$ and substrating $\dfrac{1}{2}\eta^2\left[K*P_n(t,T_i)+I(t)P_r(t,T_i)\right]$ we obtain :

$$\frac{IIC(t,T,I,N)*\eta}{N} - \frac{1}{2}\eta^2\left[K*P_n(t,T_i)+I(t)P_r(t,T_i)\right] = I(t)P_r(t,T_i)\log\left(\frac{\Gamma_i(t)}{K}\right) - K*P_n(t,T_i)\log\left(\frac{\Gamma_i(t)}{K}\right) \tag{20}$$

Factoring by $\log\left(\dfrac{\Gamma_i(t)}{K}\right)$,

$$\frac{IIC(t,T,I,N)*\eta}{N} - \frac{1}{2}\eta^2\left[K*P_n(t,T_i)+I(t)P_r(t,T_i)\right] = \log\left(\frac{\Gamma_i(t)}{K}\right)\left[I(t)P_r(t,T_i)-K*P_n(t,T_i)\right] \tag{21}$$

Dividing by $\left[I(t)P_r(t,T_i)-K*P_n(t,T_i)\right]$,

$$\log\left(\frac{\Gamma_i(t)}{K}\right) = \frac{\dfrac{IIC(t,T,I,N)*\eta}{N} - \dfrac{1}{2}\eta^2\left[K*P_n(t,T_i)+I(t)P_r(t,T_i)\right]}{\left[I(t)P_r(t,T_i)-K*P_n(t,T_i)\right]} \tag{22}$$

Multiplying the second term by $\dfrac{I(t)P_r(t,T_i)-K*P_n(t,T_i)}{I(t)P_r(t,T_i)-K*P_n(t,T_i)}$,

$$\log\left(\frac{\Gamma_i(t)}{K}\right) = \frac{IIC(t,T,I,N)*\eta}{N\left[I(t)P_r(t,T_i)-K*P_n(t,T_i)\right]} + \frac{1}{2}\eta^2\frac{\left(K*P_n(t,T_i)\right)^2-\left(I(t)P_r(t,T_i)\right)^2}{\left[I(t)P_r(t,T_i)-K*P_n(t,T_i)\right]^2} \tag{23}$$

Power of 10,

$$\Gamma_i(t) = K * 10^{\frac{IIC(t,T,I,N)*\eta}{N\left[I(t)P_r(t,T_i) - K*P_n(t,T_i)\right]} + \frac{1}{2}\eta^2 \frac{(K*P_n(t,T_i))^2 - (I(t)P_r(t,T_i))^2}{\left[I(t)P_r(t,T_i) - K*P_n(t,T_i)\right]^2}} \tag{24}$$

We have calculated the Forward Inflation CPI using again the FCA method by Jarrow and Yildirim (2003). We must take attention that the denominator of the fraction doesn't become equal to zero. Therefore our formula will not be defined.

4 Results

4.1 Inflation rate

We will present a numerical application of our theoretical methods using the Euro-Zone harmonized index of consumer prices excluding tobacco (HICP-XT) of September 4th 2008 (refers to the month of June because of the lag). Like Mercurio and Moreni (2009)[**?**] :

We report in Table 1 the ZC swap rates, the forward CPI's $\Gamma_i(0)$, the forward CPI ratios $Y_i(0)$, the YY swap rates computed under the assumption of no-drift adjustment ($D \approx 0$), the YY swap rates quoted by a broker and the corresponding YY forward inflation rates $Y_i(0)$.

T_i (Y)	Mkt ZCS	$\Gamma_i(0)$	$\mathbf{Y}_i(0)$	Implied YYS	Mkt YYS	**Mkt implied**
1	1.865%	110.6	**1.865%**	1.865%	1.865%	**1.865%**
2	2.190%	113.3	**2.516%**	2.197%	2.190%	**2.528%**
3	2.280%	116.1	**2.460%**	2.284%	2.275%	**2.458%**
4	2.335%	119.0	**2.500%**	2.336%	2.330%	**2.510%**
5	2.370%	122.0	**2.510%**	2.368%	2.364%	**2.516%**

We have used the Moreni's calibration :

It follows a SABR model for the calibration. The SABR model is a stochastic volatility model, which attempts to capture the volatility smile in derivatives markets. The name stands for "Stochastic Alpha, Beta, Rho", referring to the parameters of the model. The SABR model is widely used by practitioners in the financial industry, especially in the interest rates derivatives markets. It was developed by Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward.The SABR model describes a single forward F, such as a LIBOR forward rate, a forward swap rate, or a forward stock price. The volatility of the forward F is described by a parameter σ. SABR is a dynamic model in which both F and σ are represented by stochastic state variables whose time evolution is given by the following system of stochastic differential equations:

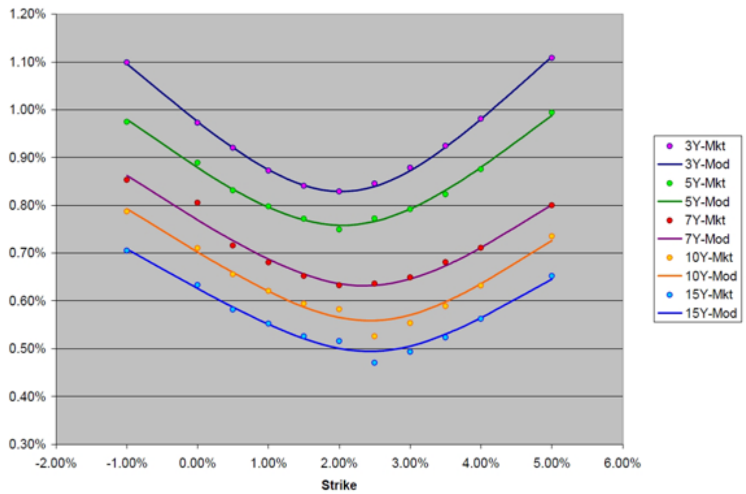

Figure 2: Calibration results: market and model implied volatilities for caplets/floorlets maturing in 3, 5, 7, 10,15 years, correlated case, model implied forwards.

$$dF_t = \sigma_t F_t^{\beta} dW_t$$

$$d\sigma_t = \alpha \sigma_t dZ_t$$

with the prescribed time zero (currently observed) values F_0 and σ_0. Here, W_t and Z_t are two correlated Wiener processes with correlation coefficient $-1 < \rho < 1$.

4.2 Taylor rule

With the Forward Inflation rates founded in section 4.1 we will assume a taylor rule for finding interest rate of the BCE or FED (depending of CPIs). We define in (8) the general formula of the Taylor rule. For the USA, Taylor define a variant of the formula but in 1999, different searchers have found parameters for the US market. Then Taylor has sorted these results in 5 rules with different pound :

$$i_t = \rho_{i_{t-1}} + g_\pi \pi_t + g_y y_t + g_0 \tag{25}$$

$$\begin{aligned} y_t &= -\beta(i_t - \pi_t - r) + u_t \\ \pi_t &= \pi_{t-1} + \alpha . y_{t-1} + e_t \end{aligned} \tag{26}$$

Where y_t is the deviation of real and forward GPB's ratio, i_t is the nominal interest rate, π_t is the inflation rate, e_t et u_t are 2 Brownian motions, a, b, r are parameters and rules parameters are g_π, g_y, g_0

Rule	g_π	g_y	ρ
Rule type 1	3	0.8	1
Rule type 2	1.2	1	1
Rule type 3	1.5	0.5	0
Rule type 4	1.5	1	0
Rule type 5	1.2	0.06	1.3

Some rules let inflation more meaning than GPB. Rule 3 is nearly the same that Taylor define in 1993. But we will not use this rule to compare inflation rate found with the economic methods and financial because of the lake of data. Actually, we will not use Taylor rule to test our result because we do not have data for GPB. Thus we will prefer use the New Keynesian Phillips curve.

4.3 Monetary policy : New Keynesian Phillips Curves

The Phillips curve has been a central topic in macroeconomics since the 1950s and its successes and failures have been a major element in the evolution over time of the discipline. We will now discuss how a popular modern version of the Phillips curve, known as the "New Keynesian" Phillips curve, that is consistent with rational expectations. Modern Keynesian thought – on which the assumed efficacy of monetary policy rests – still proposes a short-run Phillips curve based on the idea that prices (or at least aggregate prices) are "sticky."

The idea that there should be some sort of positive relationship between inflation and output has been around almost as long as economics itself, but the modern incarnation of this relationship is usually traced to a late 1950s study by the LSE's A.W. Phillips, which documented a statistical relationship between wage inflation and unemployment in the UK. This "Phillips Curve" relationship was then also found to work well for price inflation and for other economies, and it became a key part of the standard Keynesian textbook model of the 1960s. As Keynesian economists saw it, the Phillips curve provided a menu of tradeoffs for policy-makers: They could use demand management policies to increase output and decrease unemployment, but this could only be done at the expense of higher inflation.

The New Keynesian Phillips Curves[**?**] (NKPC) generally looks like this:

$$\pi_t = \beta.E_t\left\{\pi_{t+1}\right\} + \kappa.x_t \tag{27}$$

$$x_t = y_t - y_t^f$$

$$E_t\{\pi_{t+1}\} = \frac{1}{\beta}[\pi_t - \kappa.x_t] \tag{28}$$

Where π_t is the inflation rate at time t, $E\{\pi_{t+1}\}$ is the forward inflation rate for time $t+1$, x_t is the output gap, κ and β and are parameters. Notice that (unlike the original Phillips curve), it is forward looking. There are criticisms of the NKPC, but they are mostly about how it is derived rather than its existence. Here we have used these calibrations[**?**] :

Area/Var	β	κ	x_t
Euro Area calibration	0.843	0.0705	0.2
Woodford calibration	0.99	0.01	0.2

We will try to verify our forward inflation rate with the NKPC and Forward Inflation Rates found in this research[**?**]. Using these both calibration we find :

Year	$E_t\{\pi_{t+1}\}$ 1st calib.	$E_t\{\pi_{t+1}\}$ 2nd calib.	π_{t+1}	π_t
2008	1.7%	1.4%	2.53%	1.6%
2009	0.4%	0.4%	2.46%	-0.2%

As there result are based on a YYO/ZCO established in Sept. 2008 we do not have access to real inflation rate after 2009. Therefore we can not compute a complete analysis of our results because of the lake of data. But we can see a difference of 1,1 between the $E_t\{\pi_{t+1}\}$ 2nd calibration and the calculated Forward Inflation Rate and 0,8 with the first calibration.

When we will use the hybrid NK formula to complete our work.

$$\pi_t = \beta.E_t\{\pi_{t+1}\} + \kappa.x_t + (1-\beta).\pi_{t-1} \tag{29}$$

$$x_t = y_t - y_t^f$$

$$E_t\{\pi_{t+1}\} = \frac{1}{\beta}[\pi_t - \kappa.x_t - (1-\beta).\pi_{t-1}] \tag{30}$$

Where π_t is the inflation rate at time t, π_{t-1} is the inflation rate at time $t-1$, $E\{\pi_{t+1}\}$ is the forward inflation rate for time $t+1$, x_t is the output gap, κ and β and are parameters.

Year	$E_t\{\pi_{t+1}\}$ 1st calib.	$E_t\{\pi_{t+1}\}$ 2nd calib.	π_{t+1}	π_{t-1}	π_t
2008	1.12%	1.38%	2.5%	3.1%	1.6%

The result of the 2nd calibration is barely the same that with the NK simple method. The difference with the 1st calibration is worst than with the NK standard method.

5 Interpretations

For the equation finance models, we have neglected the time lags that are actually present in the contract definition of the derivatives (ie: swaps) to simplify things. We consider that the CPI index is set at the maturity date and not three months earlier, as in typical market contracts. We have used the numerical results of Moreni and Mercurio's (2009) working paper. The calculus of Forward Inflation Rate was very relevant with the Inflation Rate estimated by economics. That is why we choose these results because we could not find enough data from Bloomberg and the Internet.

The lag not considered in our financial models can be a justification of the different values of estimated Forward Inflation Rate and real Inflation Rate.

The inflation rates used in NKPC are for 2008 one of the global year whereas for 2009 it is the one known in September 2009 estimated for the year 2009. We have to know that the calibration used for the NKPC models are calibrations took from others papers but not estimated especially for 2009. That is why, it can explain a little difference between a calibration made for 2009 and one's we have used.

6 Conclusion

Our paper deals with the calculating of the forward inflation rate with finance formulae of Inflation Indexed Derivatives. the first step in this research was to find documentation and study what has been done on this topic. Pricing of Inflation Indexed Derivatives can be calculated using various methods such as HJM framework or market models. We decided to define 4 equations for calculating the forward inflation rate from Zero-Coupon, Year-on-Year, Swaps, TIPS European Call. These formulae use numerous parameters that are difficult to find in standard tools like Bloomberg or official state databases. The lake of data can not let us make numerical application or our formulae. The HJM framework has been used for numerous formulae. This model is weel know to be hard to calibrate. That is why, we used the data from Mercurio and Moreni (2009) calculated with their own calibration for the YY/ZC Swap (it does not use the HJM model). It confirms that Forward Inflation Rate can also be calculated using pricing formulae and not only economics surveys. We would like to use a Taylor Rule but as we did not study GPB in our research we had prefered choose the NKPC. But the New Keynesian standard and hybrid models have not prove if our model is efficient. It can be explained because in one hand because the lag where not considered in our financial models whereas in the NKPC the data takes care of it. In the other hand, the calibration of the model should have been re-estimated after the 2007-2009 crisis. But we do not have test the Zero-Coupon Swap model and the European Call of the TIPS because we did not have enough data.

With the enough data this research could be useful for Financial services or Quant who want to have another approach of Forward Inflation Rate. Our models depend of market data and only CPI. It could give finance professionnal a powerful tool for estimated the market and it makes a link between the economic and the finance world. It would be interessant to use the pricing formulae of derivatives with a market model instead of HJM framework which is more difficult to calibrate and find useful data.

7 Complement

For Figure 1. we have used the formula at t_0 from equation (10)

$$ZCIIS(0; T_M, K) = N\left(P_r(t, T_M) - P_n(t, T_M)(1+K)^M\right) \tag{31}$$

The following table contains the market data for nominal discount factors in the second columns, get the real discount factors from (31) in the third column and zero coupon swap rate in the fourth.

T_i (year)	$P_n(0,T)$	$P_r(0,T)$	ZCIIS rate (%)
1	0.97701	0.99764	2.11154
2	0.94982	0.99183	2.18754
3	0.91835	0.98145	2.23979
4	0.88433	0.96769	2.27759
5	0.84862	0.95045	2.29236
6	0.81179	0.93046	2.30001
7	0.7746	0.90887	2.30992
8	0.73785	0.88644	2.31991
9	0.70218	0.86254	2.32495
10	0.66773	0.84109	2.33499

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